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LETTER TO THE EDITOR

On the spectrum of the Dirichlet Laplacian on broken strips

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Abstract
We investigate the spectrum of the Dirichlet Laplacian on broken strips. It is shown that there is exactly one eigenvalue below the bottom of the essential spectrum for any angle corresponding to the broken strip.

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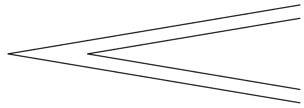
1. Introduction

The behaviour of the spectrum of the Dirichlet Laplacian in thin domains has attracted the attention of many specialists; see the recent survey by Kuchment [6] and the series of works by Exner and his collaborators [2, 3] and Avishai *et al* [1].

In this letter we consider the geometric object which is usually referred to as a broken strip (BS). The BS is the neighbourhood of the angle formed by the rays $\phi = 0, \phi = 2\alpha$, $0 < \alpha < \pi/2$ (in polar coordinates). The problem is what should be taken as the neighbourhood.

Many authors ([1, 4], etc.) have taken the neighbourhood to be made of two half-strips of equal width intersecting at angle 2α.



Exner *et al* [4] proved that at least one lower eigenvalue does always exist for $\alpha = \pi/4$. Later, Avishai *et al* [1] have shown that for small α many eigenvalues appear, and their number tends to infinity as $\alpha \to 0$.

The reason for the appearance of many eigenvalues is the choice of the neighbourhood.

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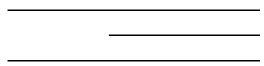
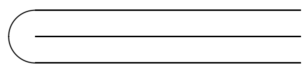
Indeed, the limiting shape of the BS as $\alpha \to 0$



gives for $-\Delta$ the threshold $\left(\frac{\pi}{2a}\right)^2$ which is smaller than $\left(\frac{\pi}{a}\right)^2$ for any particular α. Here a is the width of each half-strip.

It turns out that the additional eigenvalues do not appear for any α if we consider the smoothed neighbourhood of the angle (its geometry is explained in section 2 after theorem 2.1) which seems to be a natural analogue of the curved strips, as in [2].

Here the limiting shape of the BS is as follows:

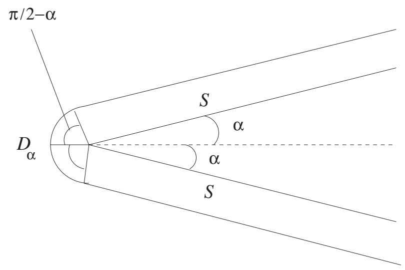


and the threshold is $\left(\frac{\pi}{a}\right)^2$, as for any $0 < \alpha < \pi/2$.

The main result of this letter yields the existence of exactly one lower eigenvalue for any such BS independently of the value of α.

2. Main result

Theorem 2.1. *Let Ω_α be the union of two infinite semi-strips S of width $a > 0$ joined by an angular sector D_α with angle $\pi - 2\alpha, 0 < \alpha \leqslant \pi/2$. Then, for any $\alpha, 0 < \alpha \leqslant \pi/2$ the Dirichlet Laplacian in this domain has exactly one eigenvalue below the threshold $\left(\frac{\pi}{a}\right)^2$.*



Proof. Due to the scaling arguments, it is sufficient to take $a = \pi$. In this case, the threshold is 1.

Consider an auxiliary eigenvalue problem in the domain D_α:

$$-\Delta u = \lambda u \qquad u|_{r=\pi} = 0 \qquad \left.\frac{\partial u}{\partial \phi}\right|_{\phi=0} = \left.\frac{\partial u}{\partial \phi}\right|_{\phi=\pi-2\alpha} = 0. \tag{1}$$

The corresponding quadratic form is $\mathbf{a}[u] = \int_{D_\alpha} (|u_r|^2 + r^{-2}|u_\phi|^2) r \,\mathrm{d}r\,\mathrm{d}\phi$, on the domain $\mathrm{Dom}\,\mathbf{a} = \{u \in H^1(D_\alpha) : u|_{r=\pi} = 0\}$. For any $\alpha \in (0, \pi/2)$ the function $u_0 = \sin r$ lies in $\mathrm{Dom}\,\mathbf{a}$, and

$$\mathbf{a}[u_0] - \int_{D_\alpha} |u_0|^2 r \,\mathrm{d}r\,\mathrm{d}\phi = 0.$$

Expressing any function $u \in \mathrm{Dom}\,\mathbf{a}$ as $u = u_0 + v$, we easily obtain

$$\mathbf{a}[u] - \int_{D_\alpha} |u|^2 r \,\mathrm{d}r\,\mathrm{d}\phi = \mathbf{a}[v] - \int_{D_\alpha} |v|^2 r \,\mathrm{d}r\,\mathrm{d}\phi - 2\int_{D_\alpha} \mathrm{Re}\, v \cos r \,\mathrm{d}r\,\mathrm{d}\phi. \tag{2}$$

The uniqueness of the eigenvalue below the threshold immediately follows from here, since $\mathbf{a}[u] - \int_{D_\alpha} |u|^2 \geqslant 0$ for all functions $u \in \mathrm{Dom}\,\mathbf{a}$ satisfying the orthogonality condition

$$\int_{D_\alpha} u \cos r \,\mathrm{d}r\,\mathrm{d}\phi = 0 \tag{3}$$

(note that u_0 meets this condition). Therefore,

$$\int_{\Omega_\alpha} (|\nabla u|^2 - |u|^2)\,\mathrm{d}x\,\mathrm{d}y \geqslant 0$$

for all $u \in H^{1,0}(\Omega_\alpha)$ satisfying the same orthogonality condition (3).

Now we turn to the existence. Since u_0 does not satisfy equation (1), it is not an eigenfunction of our problem, and therefore it is not a minimizer of its quadratic form. Therefore, there exists a function $\widetilde{u} = u_0 + v \in \mathrm{Dom}\,\mathbf{a}$, such that

$$\mathbf{a}[\widetilde{u}] < \int_{D_\alpha} |\widetilde{u}|^2 r \,\mathrm{d}r\,\mathrm{d}\phi.$$

There are many ways to choose such a function $\widetilde{u}$. One of the possible choices corresponds to $v = Ar\cos^2 r \sin\phi$, with a small enough, positive constant A. This can be easily checked with the help of equality (2). The function $\widetilde{u}$ is extended to the whole of D_α by reflection.

As soon as the function $\widetilde{u}$ is chosen, we extend it to the half-strip $S = \{(x, y) : x > 0, 0 < y < \pi\}$ (this time, in the Cartesian coordinates) $\widetilde{u}(x, y) = \mathrm{e}^{-kx}\sin y$. In a similar way, we extend the function to the rest of Ω_α.

The function constructed lies in $H^1(\Omega_\alpha)$, it is equal to zero on the boundary, and

$$\int_{\Omega_\alpha} (|\nabla\widetilde{u}|^2 - |\widetilde{u}|^2)\,\mathrm{d}x\,\mathrm{d}y = \int_{D_\alpha} (|\nabla\widetilde{u}|^2 - |\widetilde{u}|^2)\,\mathrm{d}x\,\mathrm{d}y + \pi k/2.$$

Choosing k small enough, we obtain $\int_{\Omega_\alpha} (|\nabla\widetilde{u}|^2 - |\widetilde{u}|^2)\,\mathrm{d}x\,\mathrm{d}y < 0$. □

Remark 2.2. Note that Ω_α is the one-sided neighbourhood of the angle. It is interesting to note that for the two-sided neighbourhood of the angle the result fails; for small α there are many eigenvalues below the threshold. This can be shown using the same argument as that used in [1].

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